WHAT DO WE DO TODAY?!

- Systems of Non-Linear Equations
- Multidimensional Interpolation

SYSTEM OF NONLINEAR EQNS

Iterative Procedure: Based on …

… Taylor Series expansions (what else!)

Consider the following two equation system in two unknowns x and y.

$$\begin{aligned} f(x, y) &= 0 \\ g(x, y) &= 0 \end{aligned}$$

Idea: Build two sequences $x_0, x_1, \ldots$ and $y_0, y_1, \ldots$ that converge to the roots.

Approach: Expand $f(x_{n+1}, y_{n+1})$ and $g(x_{n+1}, y_{n+1})$ about the point (x_n, y_n) using Taylor series.

NON-LINEAR EQNS ...

$$f(x_{n+1}, y_{n+1}) = f(x_n, y_n) + (x_{n+1} - x_n) f_x(x_n, y_n) + \\ (y_{n+1} - y_n) f_y(x_n, y_n) + \cdots$$

and

$$g(x_{n+1}, y_{n+1}) = g(x_n, y_n) + (x_{n+1} - x_n) g_x(x_n, y_n) + \\ (y_{n+1} - y_n) g_y(x_n, y_n) + \cdots$$

where

$$f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}, g_x = \frac{\partial g}{\partial x}, g_y = \frac{\partial g}{\partial y}$$

are the partial derivatives

NEWTON'S METHOD

Newton's Method: From the Taylor series expansion

- $f(x_{n+1}, y_{n+1}) = 0, \quad g(x_{n+1}, y_{n+1}) = 0$
- Drop all higher-order terms other than linear

Define

$$\Delta x_n = x_{n+1} - x_n$$
$$\Delta y_n = y_{n+1} - y_n$$

and Δx_n and Δy_n are solutions of the system of linear equations

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} \Delta x_n \\ \Delta y_n \end{pmatrix} = \begin{pmatrix} -f \\ -g \end{pmatrix}$$

- All quantities are evaluated at (x_n, y_n)
- Iterate

$$\begin{aligned} x_{n+1} &= x_n + \Delta x_n \\ y_{n+1} &= y_n + \Delta y_n \end{aligned}$$

until convergence

k–D INTERPOLATION

Let us look at interpolating a two-dimensional function

$$z = f(x, y)$$

f is unknown but its values

$$z_{ij} = f(x_i, y_j) \qquad i = 0, 1, \ldots, N, \quad j = 0, 1, \ldots, N$$

are given. We need to find $z_{uv} = f(x_u, y_v)$.

Idea: Split into *two* interpolations–on X-axis and on Y-axis.

How?

- Interpolate $n + 1$ points $(x_0, y_v), (x_1, y_v), \ldots, (x_N, y_v)$
- Interpolate the single point (x_u, y_v)

PROBLEMS

1. Solve:

$$\begin{aligned} 3(y-x) &= 0 \\ 2x - y - xz &= 0 \\ xy - z &= 0 \end{aligned}$$

2. Interpolate the value at $(2.5, -1.5)$ given

f(-2 , -1) = 16.916, f(-2 , 0) = 15.04, f(-2 , 1) = 13.164
f(-2 , 2) = 11.288, f(-1 , -1) = 5.636, f(-1 , 0) = 3.76
f(-1 , 1) = 1.884, f(-1 , 2) = 0.008, f(0 , -1) = 1.876
f(0 , 0) = 0.0, f(0 , 1) = -1.876, f(0 , 2) = -3.752
f(1 , -1) = 5.636, f(1 , 0) = 3.76, f(1 , 1) = 1.884
f(1 , 2) = 0.008, f(2 , -1) = 16.916, f(2 , 0) = 15.04
f(2 , 1) = 13.164, f(2 , 2) = 11.288