ENGINEERING STAFF COLLEGE OF INDIA

DIGITAL IMAGE PROCESSING
WAVELETS

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OVERVIEW

- Introduction
  - . . . in Pictures
  - . . . in Detail and Rigour
- Wavelets in Image Processing
  - Multi-resolution analysis
  - Compression
  - Texture analysis, esp., segmentation
  - Watermarking and Steganography
  - Many, Many More . . .
- Conclusions

AN IMPORTANT QUESTION

- Given a signal (or an image), find out which frequencies are present and **find out where they are located**
  - Fourier analysis finds out the frequencies present but does not give any indication about their locations
- What do we need?
  - A set of functions that represent different frequencies (in case of Fourier analysis, they are sines and cosines of various frequencies)
  - Functions with a **finite extent** (sines and cosines have infinite extent) — location is identified by a translation parameter
- Where do we need such a decomposition?
  - texture segmentation
  - object identification
  - clustering

A SECOND QUESTION

It is always possible to go from a high-resolution signal to a low-resolution signal, but . . .

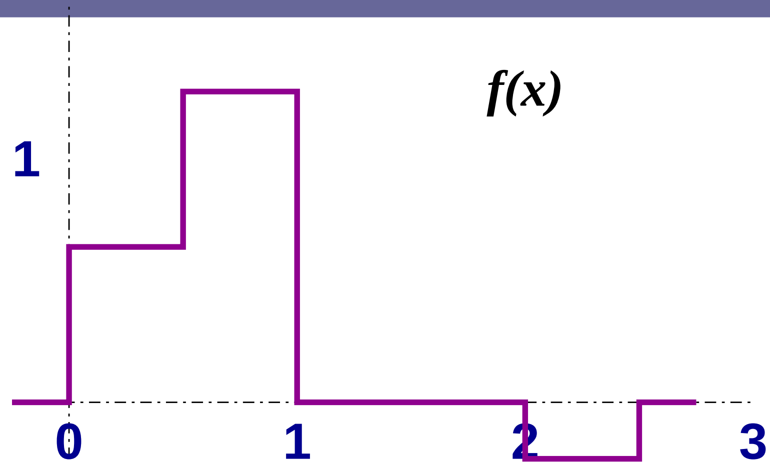
. . . is it possible to go from a low-resolution to a high-resolution signal?

- One possible solution: represent efficiently the information lost in the low-resolution image and add it later

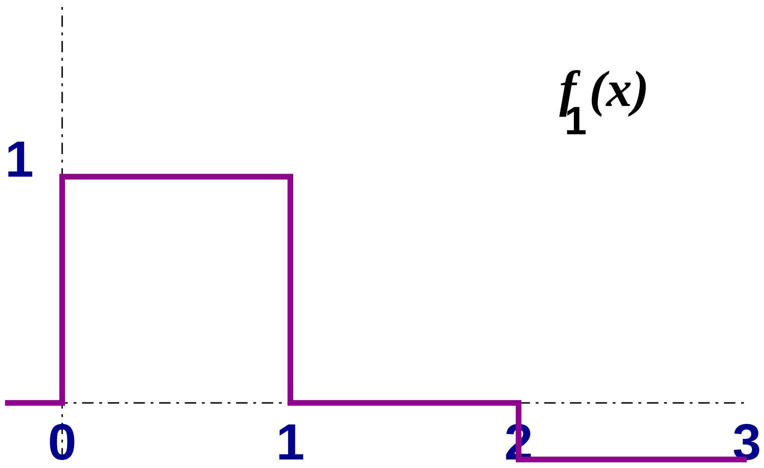
So, the question is again one of representation!

. . . in Pictures

AN EXAMPLE

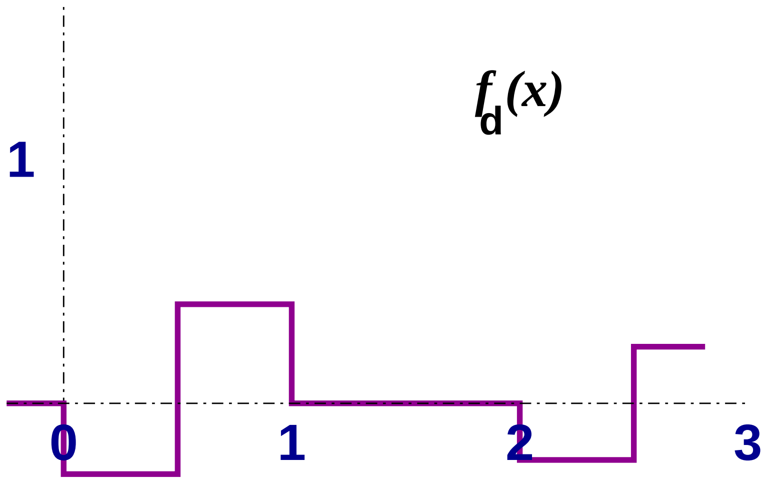


Original Function



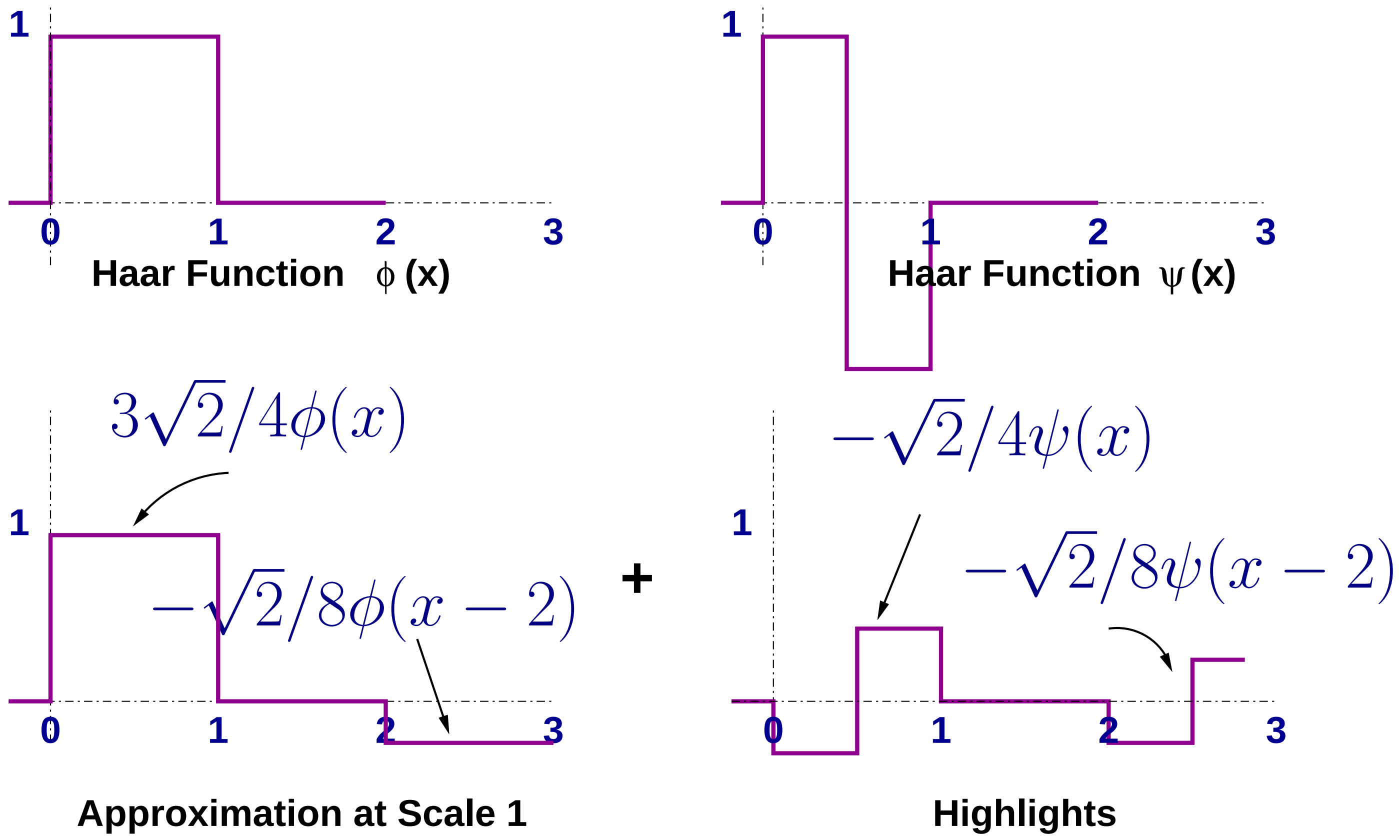
Approximation at Scale 1

+



Highlights

HAAR WAVELETS

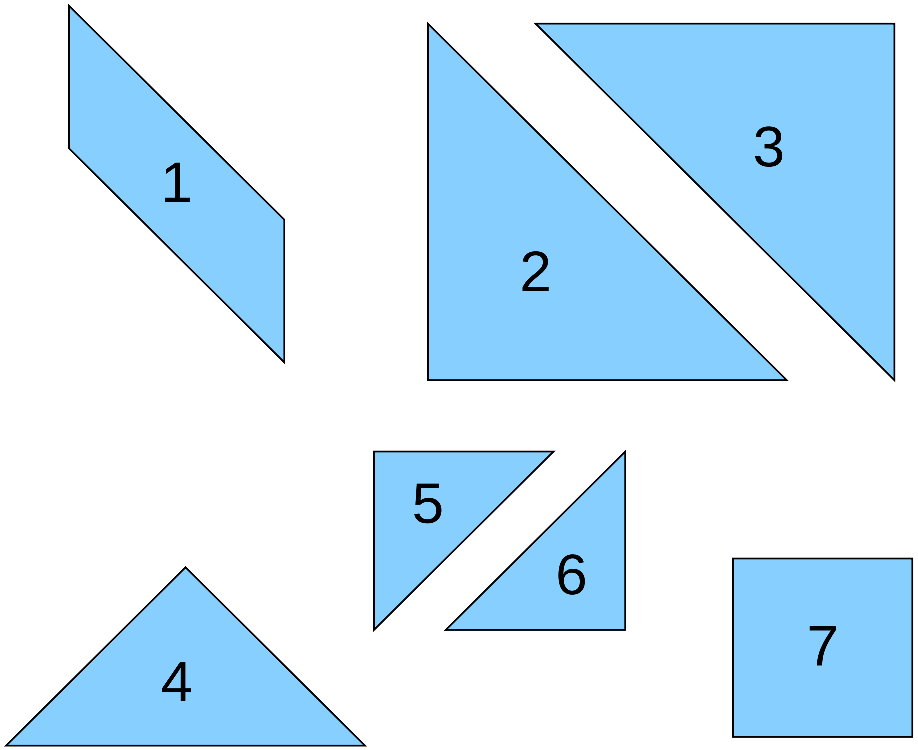


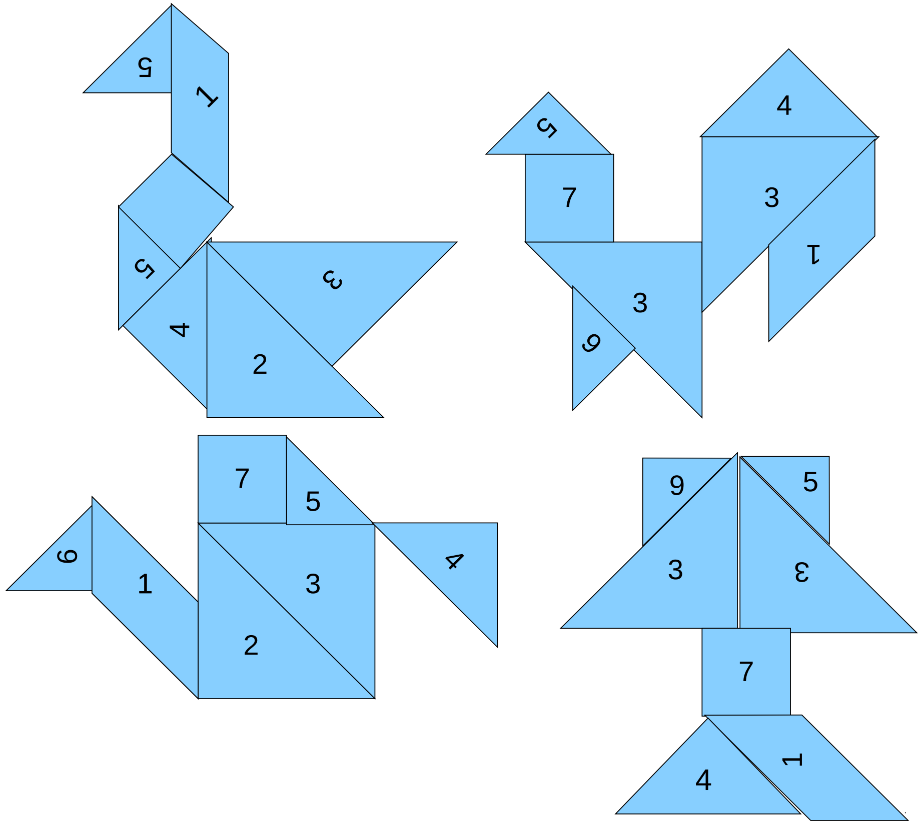
$$f(x) = 3\frac{\sqrt{2}}{4}\phi(x) - \frac{\sqrt{2}}{8}\phi(x-2) - \frac{\sqrt{2}}{4}\psi(x) - \frac{\sqrt{2}}{8}\psi(x-2)$$

. . . in Detail and Rigour

TANGRAMS

Tangrams is an ancient Chinese game where a player makes a variety of shapes using 7 standard pieces





BASIS FUNCTIONS AND COMBINATIONS

Now, let us go to mathematics and formalize the concepts in tangrams.

- Each piece is thought of as a function, $\phi_k(x)$
- Each shape is a combination of such functions

Any arbitrary function may be spatially decomposed as

$$f(x) = \sum_k \alpha_k \phi_k(x)$$

where α_k and $\phi_k(x)$ are real-valued coefficients and functions.
If the expansion is unique, i.e., there is only one set of α_k for any given function $f(x)$, then $\{\phi_k(x)\}$ forms a basis and $\phi_k(x)$ is a basis function.
Tangram puzzle is all about finding α_k that represent position and rotation angle through which each $\phi_k(x)$ should be transformed for forming a given shape!

SPANS AND SPACES

- The set of functions that can be expressed as a linear combination of basis functions is called the span of the basis functions, and they form a space V denoted by

$$V = \overline{\text{Span}_k\{\phi_k(x)\}}$$

- $f(x) \in V$ means that $f(x)$ is expressible as a linear combination of basis functions $\phi_k(x)$

In tangrams, span is the set of all shapes that can be made with the 7 pieces. An example of a shape that is not in such a span or does not belong to the space of shapes is a circle.

ORTHONORMAL BASIS

There also exist a set of dual functions $\hat{\phi}_k(x)$ such that

$$\alpha_k = \int \hat{\phi}_k^*(x) f(x) dx \tag{1}$$

where $*$ indicates the complex conjugate. Real-valued functions are their own complex conjugates.

If the basis functions are orthonormal, i.e.,

$$\phi_j(x) \cdot \phi_k(x) = \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases}$$

then $\phi_k(x)$ is its own dual.

We can see that Fourier Series is a special case of orthonormal expansions. Fourier series gets the frequency interpretation because its basis functions $e^{j2\pi ux}$ represent sinusoids through the well-known equation $e^{jx} = \cos x + j \sin x$.

SCALING FUNCTIONS

Let us now talk about choosing good $\phi_k(x)$.
Define a family of scaled, integer translated versions of a single variable function, $\phi(x)$, as

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$$

- How many basis functions are there in tangrams?
  - Three (Square, trapezium and triangle)
  - Five scaled versions of triangle — two small triangles, one scaled by a factor of 2, and two others scaled by a factor of 4 each
- The space spanned by scaling functions for a particular scale coefficient $j = j_0$ is denoted by V_0.

We find that the functions get finer and finer as j increases. This forms the foundation of multiresolution analysis.

MALLAT'S SCALING CRITERIA

1. Scaling function is orthogonal to its integer translates
2. The subspaces spanned for different j should be nested

$$V_{-\infty} \subset \ldots \subset V_0 \subset V_1 \ldots \subset V_\infty$$

3. The only function common to all subspaces is $f(x) = 0$
4. Any function can be represented with arbitrary precision

If these criteria are met, then a function at lower resolution j may be represented by a function at a higher resolution as

$$\begin{aligned} \phi_{j,k}(x) &= \sum_n h_\phi(n)\phi_{j+1,n}(x) \\ &= \sum_n h_\phi(n)2^{(j+1)/2}\phi(2^{j+1}x - n) \end{aligned}$$

since $\phi(x) = \phi_{0,0}(x)$,

$$\phi(x) = \sum_n h_\phi(n)\sqrt{2}\phi(2x - n)$$

$h_\phi(n)$ are called scaling function coefficients.

WAVELET FUNCTIONS

Wavelet functions form the basis for the difference space between two adjacently scaled subspaces

Let W_0 define the difference between V_j and V_{j+1}. If the set of functions,

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j - k)$$

form the basis for W_0, then the functions $\psi_{j,k}(x)$ are called wavelet functions and

$$V_{j+1} = V_j \oplus W_j$$

If $L^2(R)$ represents the space spanned by all the square-integrable real-valued functions, then it is possible to write

$$L^2(R) = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots$$

WAVELET FUNCTIONS. . .

The previous equation may be generalized to yield

$$L^2(R) = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus \dots$$

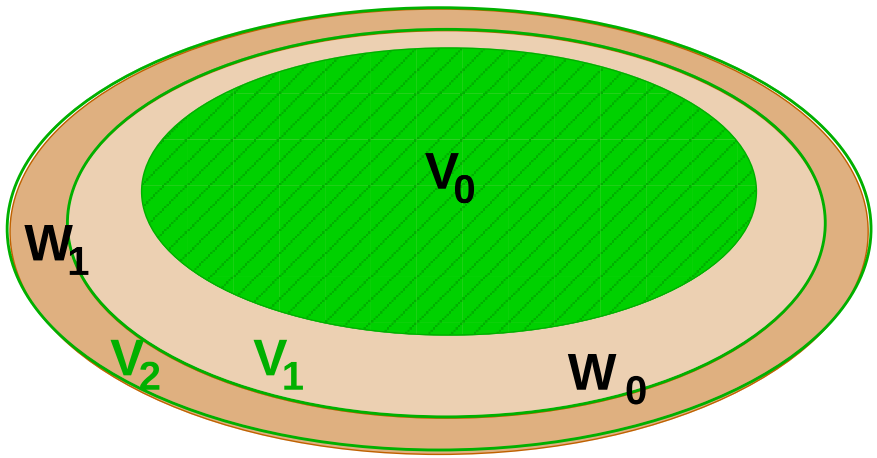
where j_0 is an arbitrary starting scale or starting resolution.

- Wavelet functions may be scaled and integer translated from a single basic wavelet as

$$\psi(x) = \Sigma\, h_\psi(n)\sqrt{2}\phi(2x - n)$$

- Wavelets may also be obtained from scaling functions of the subspaces V_j

$$h_\psi(n) = (-1)^n h_\phi(1 - n)$$



WAVELET TRANSFORMS

Take a look at our multiresolution equation again

$$L^2(R) = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus \dots$$

Thus, any arbitrary function $f(x)$ may be expressed as

$$f(x) = \sum_k c_{j_0}(k)\phi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_k d_j(k)\psi_{j,k}(x)$$

$c_{j_0}(k)$ are called the approximation or scaling coefficients and $d_j(k)$ are called the wavelet coefficients.
If orthonormal basis functions are used, then

$$c_{j_0}(k) = \frac{1}{\sqrt{M}} \sum_x f(x)\phi_{j_0,k}(x)dx$$

$$d_j(k) = \frac{1}{\sqrt{M}} \sum f(x)\psi_{j,k}(x)dx$$

PUTTING IT ALL TOGETHER

- Under wavelet transforms, any function $f(x)$ is expressed as a sum of an approximation of $f(x)$ and its highlights
- Approximation is represented as a linear combination of functions $\phi(x)$, its scaled and translated versions $\phi_{j,k}(x)$
- Highlights are represented as a linear combination of functions $\psi(x)$, its scaled and translated versions $\psi_{j,k}(x)$

Normally,

$$\phi_{j,k}(x) = 2^{\frac{j}{2}}\phi(2^j x - k)$$
$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$$

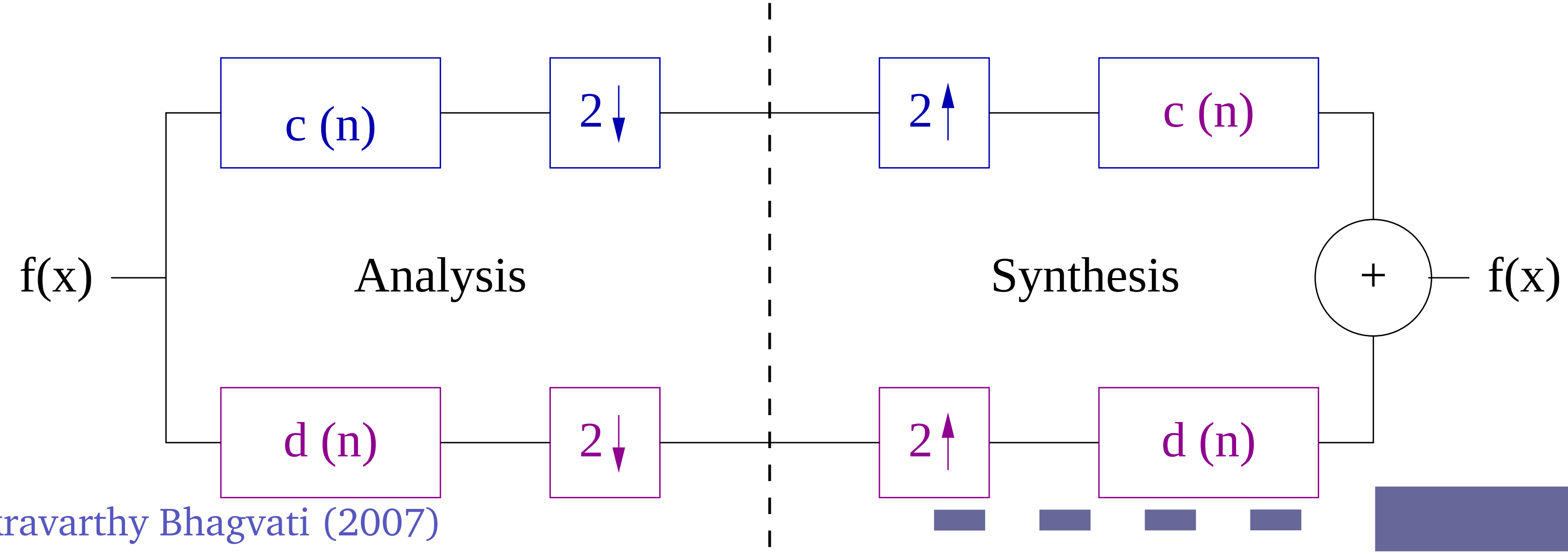
Even more beautifully, if $\phi(x)$ is chosen carefully, then $\psi(x)$ is itself derivable from $\phi(x)$
$\phi(x)$ is called the **mother wavelet**

WAVELET TRANSFORM (1-D)

- Wavelet transform is given by

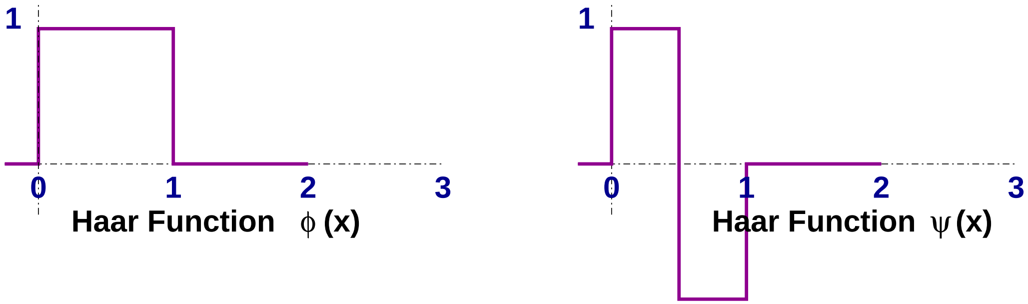
$$f(x) = \sum_k c_{j_0}(k)\phi_{j_0,k}(x) + \sum_{j=j_0}^{\infty}\sum_k d_j(k)\psi_{j,k}(x)$$

- The first expression is the approximation at scale j_0
- The second represents the details upto any arbitrary scale j
- $c_{j_0}(k)$ are the scaling or approximation coefficients
- $d_j(k)$ are the wavelet coefficients

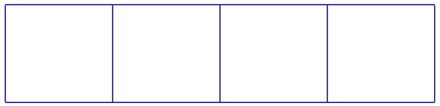


SIMPLE EXAMPLE USING HAAR WAVELETS

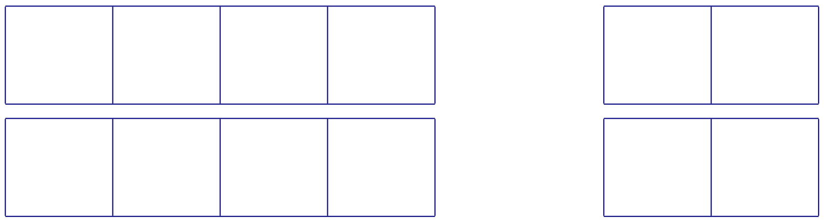
- Haar wavelets are given by



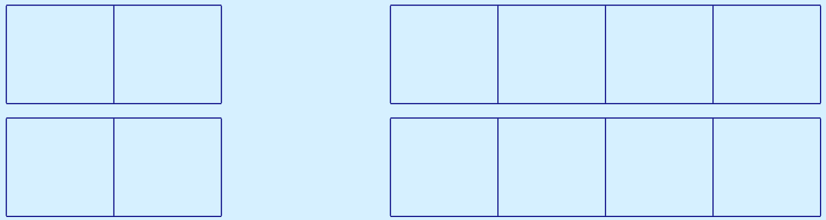
- Consider the 1-D function below



- Analysis step



- Synthesis step



WAVELET TRANSFORM IN 2-D

- The main difference from 1-D version is that the highlights come in three components – horizontal, vertical and diagonal
- Wavelet transform (2D)

$$f(x,y) = \frac{1}{\sqrt{MN}} \sum_m \sum_n c_{j_0}(m,n)\phi_{j_0,m,n}(x,y) + \frac{1}{\sqrt{MN}} \sum_{i=H,V,D} \sum_{j=j_0}^{\infty} \sum_m \sum_n d_j(m,n)\psi^i_{j,m,n}(x,y)$$

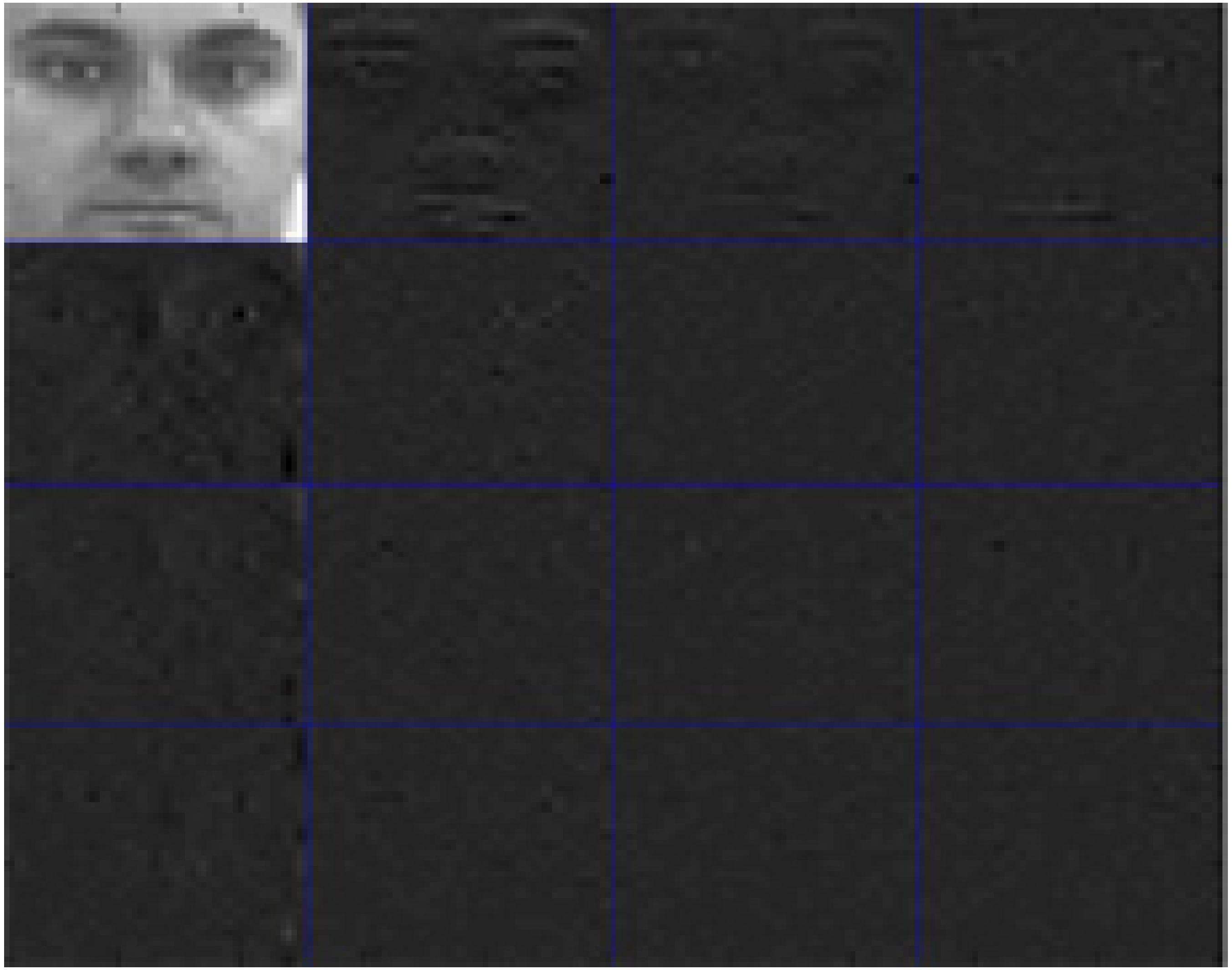
Directionally sensitive wavelets

$$\psi^H(x,y) = \psi(x)\phi(y)$$
$$\psi^V(x,y) = \phi(x)\psi(y)$$
$$\psi^D(x,y) = \psi(x)\psi(y)$$

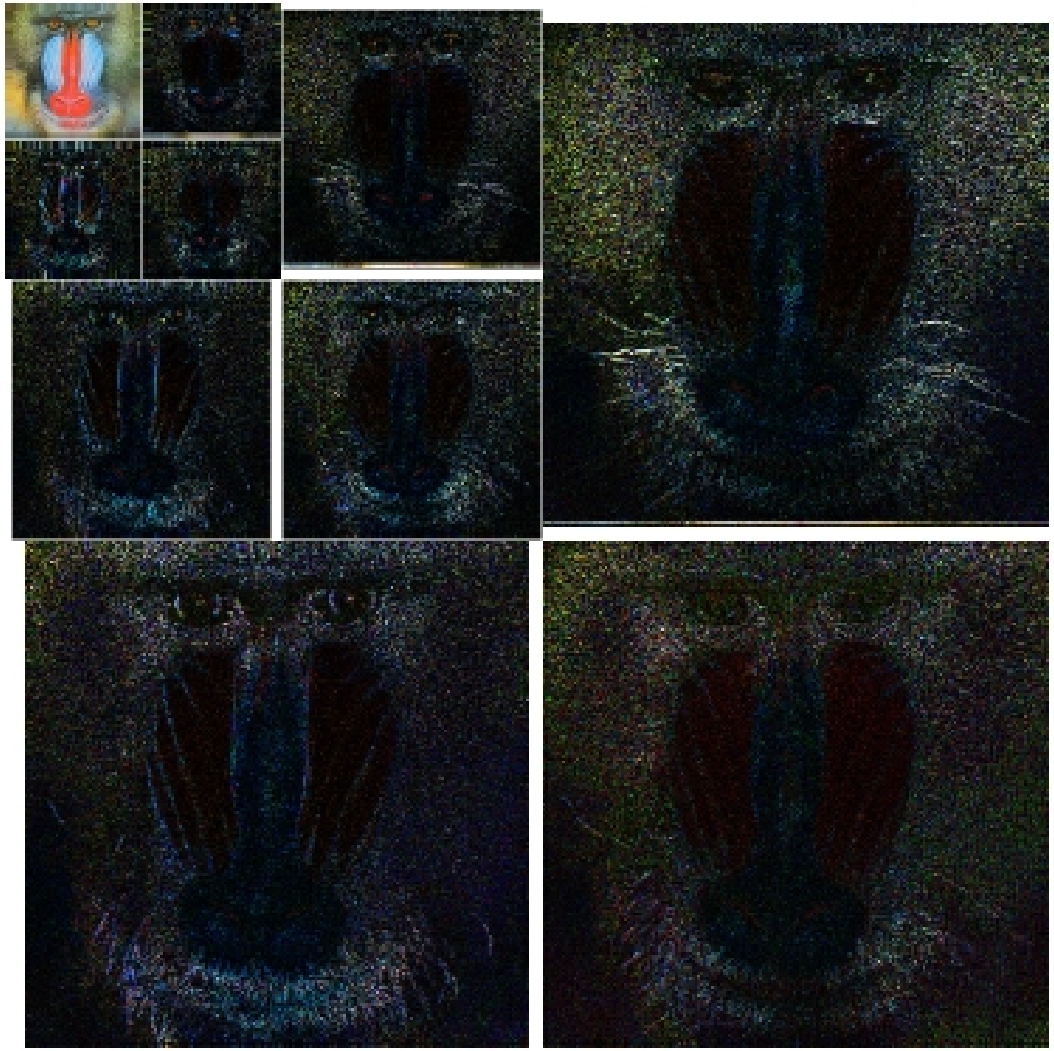
$\phi_{j_0,m,n}(x,y)$	$\psi^H_{j,m,n}(x,y)$
$\psi^V_{j,m,n}(x,y)$	$\psi^D_{j,m,n}(x,y)$

WAVELET TRANSFORM EXAMPLE

WAVELET TRANSFORM (ANOTHER EXAMPLE)



MORE EXAMPLES



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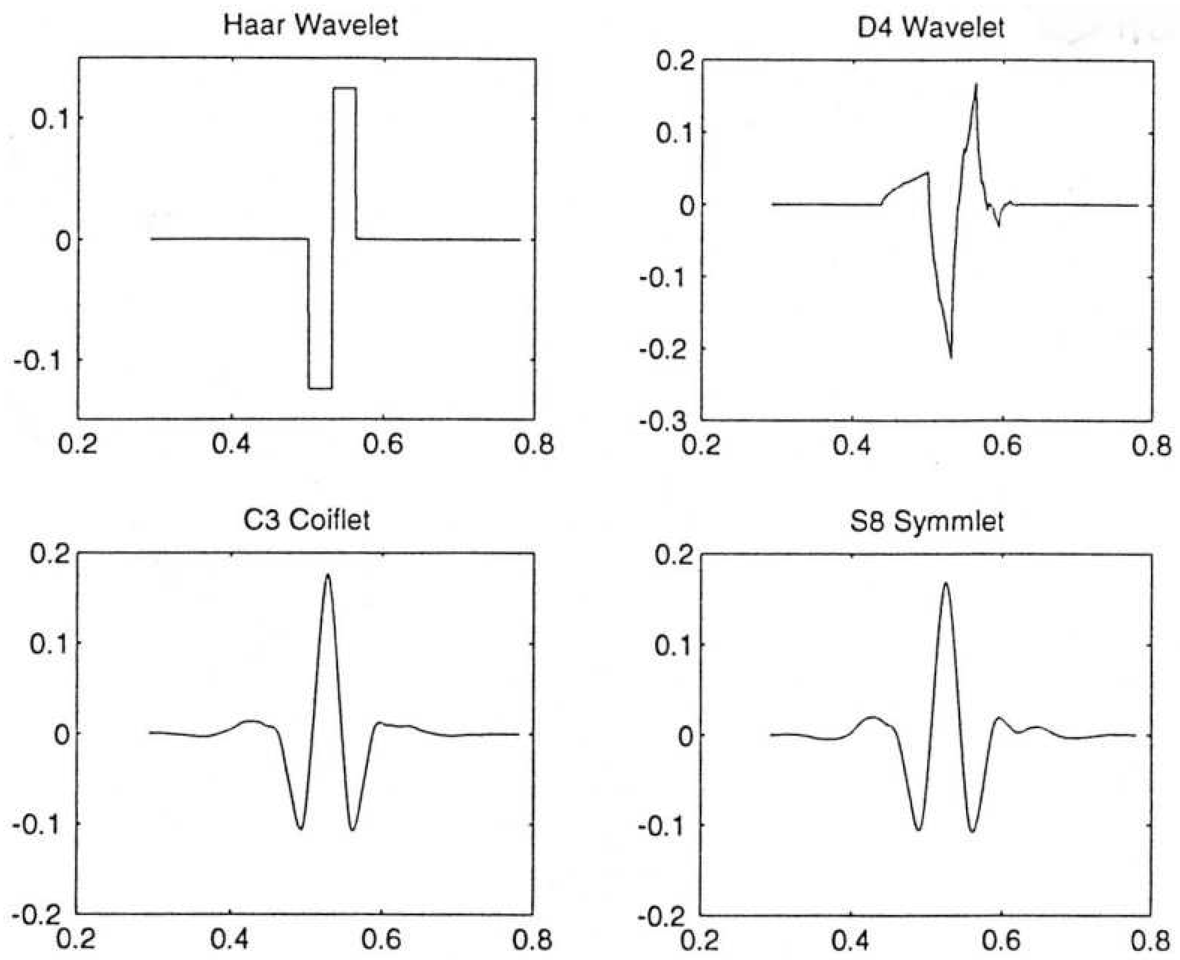

Figure 7: *Toon 0111*

CONCLUSION

- Wavelets caused a revolution in signal processing during 1990s because of their wonderful properties
- Wavelets unified a number of previous approaches such as multiresolution analysis, subband coding, Haar Transforms
- Wavelets are applied everywhere from image classification to compression to steganography and watermarking
- Essentially, wavelets once again demonstrate the power of a good, clean, rigorous and efficient representation
- Don't shun mathematics!