EPISODE 1

WHEN DOES THE WORLD END?

It is a question that has fascinated humankind from times immemorial. Remember all the hoo-ha about December 2012 and the Mayan calendar? Or the doomsday predictions of Nostradamus? Or those "world will end; all the infidels shall perish" that surface every time a comet is seen or a total solar eclipse occurs? Sure, it is a legitimate question that traditionally has been of interest to philosophers, theologians and the like, also to physicists and cosmologists. But what has it got to do with Computer Science? Why is this question here in this book?

One of the most famous problems in Computer Science, as any student of the subject will safely vouch for, is the *Tower(s) of Hanoi* problem. It was first introduced to the West by the French mathematician Édouard Lucas in 1883. He is believed to have been inspired by an Indian (referring to the country India and not the Mayans, Aztecs or any host of others!) legend, *The Towers of Brahma*.

Once upon a time in ancient India, a number of sages met and started discussing the meaning of life and when life would end. They decided to meet God Brahma, the creator, and get a definitive answer. They approached Brahma and asked him, "O Lord, can you please tell us when the world will end."

"Go back to the Earth and go to the city of Kasi (modern-day Varanasi)," said Brahma. "There, near the temple of Lord Viswanath, you will find three pillars aligned north to south. On the northernmost pillar will be 64 golden disks with the largest at the bottom and the smallest at the top in strict order of their size. Move these 64 disks to the southernmost tower obeying the following rules:

- Move *only one disk* at a time
- *Never* put a larger disk on a smaller one

The world, and therefore, all life shall end when all the disks are moved to the southernmost tower. The other tower may be used for temporary positioning of the disks. If you violate any of the two rules, the world shall be immediately destroyed."

The sages returned to Kasi and found that indeed there were three towers and disks exactly as stated by Brahma. They were a bit worried and with great reluctance and hesitation started moving the disks. We are alive today and definitely the world has not ended yet! The real question is, "How many years did Brahma give us? How much time does it take to move the disks?"

Let us attempt an analysis of the problem. Start with a *single* disk on the northernmost tower. It takes just *one move* to solve the problem. What about two disks to start with? Here is the solution (also rather straightforward).

1. Move Disk 1 to the intermediate tower

2. Move Disk 2 to the southernmost tower
3. Move Disk 1 from the intermediate to the southernmost tower

We are done in three steps. Now, what about 3 disks? Here goes …

1. Move Disk 1 to the southernmost tower
2. Move Disk 2 to the intermediate tower
3. Move Disk 1 from the southernmost to the intermediate tower
4. Move Disk 3 to the southernmost tower
5. Move Disk 1 from the intermediate to the first tower
6. Move Disk 2 from the intermediate to the southernmost tower
7. Move Disk 1 from the first to the southernmost tower

We are done in seven steps. Moving a single disk took 1 step; moving two took 3 steps; and, moving three took 7 steps. An intelligent reader can probably see the pattern now — the number of steps $S(n) = 2^n - 1$ if we represent the number of disks as n.

Let us go further: what if I did not find the pattern? Examine the solution for the three disk problem more carefully. At Step 4, notice that the largest disk, Disk 3, is not yet moved and the two smaller disks, Disks 1 and 2, are correctly placed on the intermediate tower. We moved Disk 3 to the southernmost tower, and finally, moved the two smaller disks from the intermediate to the southernmost tower. In this part, we used the first tower (that is now empty) as the intermediate tower! Aha, let that cartoon light-bulb now go off in our heads!!

> TO SOLVE A PROBLEM WITH n DISKS, WE FIRST SOLVE THE PROBLEM FOR $(n-1)$ DISKS; MOVE THE LARGEST DISK TO THE SOUTHERNMOST TOWER; AGAIN SOLVE THE PROBLEM FOR $(n-1)$ DISKS. ᵗHAT IS, WE SOLVE THE $(n-1)$ DISK PROBLEM TWICE!

This is exactly what we need to calculate the number of steps to solve the n-disk problem. Let us simply state the, as yet unknown, number of steps to solve the n-disk problem as $S(n)$. Similarly, let the number of steps required to solve the $(n-1)$-disk problem be $S(n-1)$. From the light-bulb moment, we know that $S(n)$ will consist of solving the $n-1$-disk problem, moving the n^{th} disk and then solving the $n-1$-disk problem again. So, $S(n) = S(n-1) + 1 + S(n-1)$ or

$$S(n) = 2S(n-1) + 1$$

This is a very interesting equation. It shows that the solution to the bigger problem is constructed from the solutions to a smaller problem. These are called *Recurrence Relations* in Computer Science. Recurrence relations are one of the most powerful mathematical tools in the whole of Computer Science. Where does their power come from? It comes from *unfolding* the recurrence relation, i.e., we can continue using the same relationship repeatedly. For example,

$$\begin{aligned} S(n) &= 2S(n-1)+1 \\ &= 2\{2S(n-2)+1\}+1 = 4S(n-2)+3 \\ &= 4\{2S(n-3)+1\}+3 = 8S(n-3)+7 \\ &= \ldots \end{aligned}$$

We can unfold it all the way until we get $S(n)$ in terms of $S(1)$ because we know that $S(1) = 1$. In Computer Science, often we know the answers to a smaller problem and the use of a recurrence relation helps us now solve the bigger problem.

Coming back to our question, is the above equation correct? Let us see for the 3-disk problem, i.e., $n = 3$.

$$\begin{aligned} S(3) &= 2S(2)+1 \\ &= 2\Big(2S(1)+1\Big)+1 \\ &= 2\Big(2\times 1+1\Big)+1 \\ &= 2\Big(3\Big)+1 = \mathbf{7} \end{aligned}$$

It works, at least for this case! We can see that it works for the 2-disk problem too. Not being mathematicians, who are known to ask for pesky *proofs*, we can rest easy that the equation we guessed seems correct and try to answer the original, all important question of when the world will end. But before we do that, let us do just one more thing. We can *prove* that the intelligent guess of $2^n - 1$ for an n-disk problem is correct.

Consider the problem for 5 disks. Our intelligent pattern recognising person will say the answer is $2^5 - 1$, that is 31 moves. How about the second approach?

$$\begin{aligned} S(5) &= 2S(4)+1 \\ &= 2[2S(3)+1]+1 \\ &= 2[2\{2S(2)+1\}+1]+1 \\ &= 2[2\{2(2S(1)+1)+1\}+1]+1 \\ &= 2[2\{4S(1)+2+1\}+1]+1 \\ &= 2[8S(1)+4+2+1]+1 \\ &= 16S(1)+8+4+2+1 \\ &= 16+8+4+2+1 \\ &= 2^4+2^3+2^2+2^1+2^0 = \mathbf{31} \end{aligned}$$

So, the second approach also gives the same answer. More importantly, it shows us that the intelligent guess of $2^n - 1$ is *correct*. Look at the last step above: it tells us that

$$S(n) = 2^{n-1} + 2^{n-2} + \ldots + 2^1 + 2^0$$

A very small bit of mathematics will show us that the right hand side evaluates to $2^n - 1$ which is the same as the original intelligent guess.

Now, we are ready to tackle Brahma's problem which is to compute $S(64)$. The answer, which we can now prove, is

$$2^{64} - 1 \text{ or } 18,446,744,073,709,551,615 \text{ moves}$$

If we assume that the sages can make one move every second, the world will end in 18,446, 744,073,709,551,615 seconds. How big is this number? Let us do a simple calculation. We know that there are $86,400$ seconds every day and $365.25 \times 86,400$ seconds in a year which is $31,557,600$ seconds which we can approximate to 32 million seconds. The same can be expressed as approximately 2^{25} seconds as 2^{20} is approximately one million and $2^5 = 32$. Therefore, Brahma indirectly said that the world will end in $(2^{64} - 1)/2^{25} = 2^{39}$ years. This comes out roughly to 500 billion years as 2^{30} is one billion and $2^9 = 512$.

As an interesting aside, the current age of the Earth is believed to be about 4.5 billion years and that of the Solar System about 5 billion years. The universe itself, according to the Big Bang Model in cosmology, is estimated to be 14 billion years old. More interestingly, many cosmologists believe that the universe is about 10% of the way in its life span of about 100 billion years. We, I guess, are quite safe. The universe has a long way to go!